Basic Non-Equilibrium Mechanics

Probability distributions

The moments of a probability distribution are given by the equation:

$$m_n = \langle x^n \rangle = \int_{-\infty}^{\infty} x^n p(x) \, dx$$

The zeroth moment of a probability distribution is always one. The first moment is the mean, and the variance is related to the second moment by $\sigma^2 = m_2 - m_1^2$.

We can find time evolution terms of the moments by computing integrals. In the case of a two-dimensional distribution $p(x, v, t)$ we have:

$$\frac{\partial \langle x \rangle}{\partial t} = \frac{\partial}{\partial t} \iint x p(x, v, t) \, dx \, dv = \iint x \frac{\partial p}{\partial t} \, dx \, dv$$

The characteristic function is the Fourier transform of the pdf:

$$p(k) = \int_{-\infty}^{\infty} \exp(-ikx) p(x) \, dx$$

It is useful for computing moments using the equation:

$$m_n = \langle x^n \rangle = i^n \frac{d^n p(k)}{dk^n} \bigg|_{k=0}$$

The characteristic function in term can be constructed from its moments by writing a Taylor series:

$$p(k) = \sum_{n=0}^{\infty} \frac{(-ik)^n m_n}{n!}$$

A variant of the moments are the cumulants of a pdf, which are generated by differentiating the logarithm of the characteristic function. The nth cumulant is defined as:

$$c_n = i^n \frac{d^n \log p(k)}{dk^n} \bigg|_{k=0}$$

The autocorrelation function of noise term $\xi(t)$ is defined as:

$$G(t' - t'') = \langle \xi(t') \xi(t'') \rangle = \int \xi(t') f(t')\xi(t'') f(t'') \, dt$$

The full definition in the discrete case is:

$$G(\tau) = E[(x_t - \mu)(x_{t+\tau} - \mu)]$$
Central limit theorem
If \( \{x_1, ..., x_N\} \) are independent, identically distributed random variables with mean \( \mu \) and variance \( \sigma^2 \), then the pdf of the normalised sum \( X = \frac{1}{\sqrt{N}\sigma}(\sum x_i - N\mu) \) approaches a normal distribution:

\[
p(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)
\]

If \( x = x_1 + x_2 \) is the sum of two independent random variables with pdfs \( p_1(x_1) \) and \( p_2(x_2) \), then:

\[
p(x) = \int_{-\infty}^{\infty} p_1(x_1)p_2(x_1 - x_2) \, dx_1
\]

A multivariable Gaussian distribution has the form:

\[
p(\bar{x}) = \frac{1}{(2\pi)^{n/2}|\det(\Sigma)|^{1/2}} \exp\left[-\frac{1}{2}(\bar{x} - \mu)^\top \Sigma^{-1}(\bar{x} - \mu)\right]
\]

Where \( \Sigma \) is the covariance matrix. The characteristic function takes the form:

\[
p(\bar{k}) = \exp\left(i\bar{k}^\top \mu - \frac{1}{2} \bar{k}^\top \Sigma^{-1} \bar{k}\right)
\]

For a system of \( N \) particles with some microscopic property of mean \( \bar{\mu} \) and variance \( \sigma^2 \), the derived macroscopic property \( A = \sum a_i \) has mean \( \langle A \rangle = N \bar{\mu} \) and variance \( \delta A = N^{1/2} \sigma_a \), and hence \( \frac{\delta A}{\langle A \rangle} = \frac{\sigma_a}{\bar{\mu} N^{1/2}} \).

That is, fluctuations as a proportion of the mean decline for large systems near equilibrium. This does not always occur to systems far from equilibrium, which can have \( \delta A \gg \langle A \rangle \).

Equilibrium versus non-equilibrium systems
Equilibrium systems:

- \( f_N(t, x, p) \) independent of time (this also means the system is stable; if the environment nudges it, it quickly relaxes back to a steady state)
- Isolated system: or system in contact with a heat bath
- Simple: no long-range, cooperative behaviour (except for phase transitions, which are themselves a non-equilibrium process), no evolution macroscopically (by definition)
- Entropy: increases or remains constant with time
- Internal variables are Gaussian or something like Gaussian (except at phase transitions)

Non-equilibrium systems:

- \( f_N(t, x, p) \) depends on time, so the system is in a dynamic state of change
- Open system: exchanges mass and energy with environment
- Complex: formation of macroscopic dissipative structures in space and time when huge numbers of particles cooperate over macroscopic distances. Examples include turbulent eddies in high-Reynolds-number flow, biomolecules produced in driven chemical reactions, firms created within economies, and living organisms.
- Entropy: can increase or decrease over time
- Nonlinear equations of motion for the macroscopic fields; variables are often power law
Gaussian and power law fluctuations

A system in equilibrium is expected to yield normally distributed macroscopic variables when multiple systems prepared in the same way are measured. This arises because the macroscopic energy (for example) is simply the sum of microscopic energies, so it is the sum of a very large number of iid random variables. As such the central limit theorem applies, and we expect the result to be normally distributed.

Nonequilibrium systems often exhibit power law behaviour. This is largely because non-Gaussian behaviour in the tails of the distribution requires very large N to damp out.

Entropy and the fluctuation theorem

Entropy is defined to be:

\[ S(t) = -k_B \int f_N(x, p, t) \log(f_n(x, p, t)) d^3N \]

In equilibrium this is independent of time and can be written in terms of macroscopic variables like N, V and T. In a closed system, it increases in time and approaches a maximum at equilibrium.

If we consider the infinitesimal volume of phase space around the point \((q_1, ..., q_n, p_1, ..., p_n)\), the final volume is related to the initial volume by:

\[ d\Gamma(t_f) = \exp\left[ \int_{t_i}^{t_f} \left( \frac{\partial q_i(t)}{\partial q_i(t)} + \frac{\partial p_i(t)}{\partial p_i(t)} \right) dt \right] d\Gamma(t_i) \]

This means that the phase space volume increases or decreases exponentially along trajectories.

We can consider the logarithm of the ratio of the probabilities of the same phase space trajectory being followed in the forward \(\Gamma\) and reverse \(\Gamma^*\) directions, with initial probability densities \(f\), as:

\[ \Omega_t = \log \left( \frac{f(\Gamma_0|\Gamma_0)}{f(\Gamma_t^*|\Gamma_0^*)} \right) = \log \left( \frac{f(\Omega)}{f(\Gamma_t^*)} \right) - \int_0^t \left( \frac{\partial q_i(t)}{\partial q_i(t)} + \frac{\partial p_i(t)}{\partial p_i(t)} \right) dt \]

Another way of writing this is:

\[ f(\Gamma_t^*)d\Gamma_t^* = \exp(\Omega_t) f(\Gamma_0) d\Gamma_0 \]

Using this definition, and also noting that \(\Omega_t(\Gamma_t^*) = -\Omega_t(\Gamma_0)\) we can obtain an expression for the probability that \(\Omega_t\) falls in the range \((-A, -A + dA)\) after time \(dt\):

\[ p(\Omega_t = -A) = \int \delta(\Omega_t(\Gamma_0) + A) f(\Gamma_0) d\Gamma_0 \]

\[ = \int \delta(\Omega_t(\Gamma_0^*) + A) f(\Gamma_t^*) d\Gamma_t^* \]

\[ = \int \delta(\Omega_t(\Gamma_0) + A) \exp(\Omega_t) f(\Gamma_0) d\Gamma_0 \]

\[ = \int \delta(-\Omega_t(\Gamma_0) + A) \exp(\Omega_t) f(\Gamma_0) d\Gamma_0 \]

\[ p(\Omega_t = -A) = \exp(-A) p(\Omega_t = A) \]
This is the Evans-Searles fluctuation theorem. It states that, in any given forward-reverse pair of trajectories with $\Omega_\epsilon \neq 0$, one trajectory is favoured exponentially over the other.

**The ergodic hypothesis**

The ergodic hypothesis states that during any time interval the location of the system in phase space is equally likely to be anywhere on the surface of constant energy. If the ergodic hypothesis were not true, there would be parts of phase space with the same $E$ that never see each other - i.e. a trajectory in one part would not enter the other and vice versa.

This definition means that we can compute time averaged values of some variable $f$ in a different way. The most general definition is:

$$\langle f \rangle = \frac{1}{\tau} \int_{t_0}^{t_0+\tau} f(q_i(t), p_i(t)) \, dt$$

This is usually impossible to calculate. However, under the ergodic hypothesis all locations of equal energy are equally likely, so we can simply take a weighted average over the different possible values:

$$\langle f \rangle = \int f[q_i, p_i] \rho(q_i, p_i) \, dq_i \, dp_i$$

Where $\rho(q_i, p_i)$ can be viewed as the probability density that the system will be found in phase space at $\{q_i(t), p_i(t)\}$.

**Diffusive Stochastic Processes**

**Key terminology**

A stationary process is one in which ensemble averages are independent of time. It is not the same as an equilibrium process.

In an ergodic process, the ensemble average is the same as the infinite time average:

$$\langle A(t) \rangle_{ens} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} A(t') \, dt'$$

**Random walk**

The master equation for a one-dimensional simple random walk with equal step size is given by:

$$p(x, t + \Delta t) = \int_{-\infty}^{\infty} p(x - \Delta x, t) p(\Delta x) \, d(\Delta x)$$

That is, the probability of ending up at $x$ equals the probability that you were at $x - \Delta x$ previously and made a jump of size $\Delta x$, summed over all the possible (mutually exclusive) $\Delta x$ alternatives.

To second order, and assuming $\Delta x \propto (\Delta t)^{1/2}$, we find:

$$\frac{\partial p}{\partial t} = -\langle \Delta x \rangle \frac{\partial p}{\partial x} + \frac{\langle (\Delta x)^2 \rangle}{2\Delta t} \frac{\partial^2 p}{\partial x^2}$$

**Langevin equation for Brownian motion**

The Langevin equation for the velocity of pollen grains in a fluid with the stochastic force term $\xi(t)$ is:
The stochastic term has the properties:

- In equilibrium there is no preferred motion so \( \langle \xi_i(t) \rangle = 0 \).
- If the Cartesian components of force are independent then fluctuations are given by white noise with \( \langle \xi_i(t)\xi_j(t') \rangle = 2D \delta_{ij} \delta(t-t') \).
- The parameter \( \gamma = \frac{6\pi \eta a}{m} \) with dynamic viscosity \( \eta \) and a pollen grain of mass \( m \) and radius \( a \) represents the rate of deceleration of the pollen grain.

**Fluctuation-dissipation theorem**

The theorem states that in thermal equilibrium and relatively small noise term correlation time the dampening coefficient \( \gamma \) is equal to:

\[
\gamma = \frac{m}{6k_B T} \int_{-\infty}^{\infty} \langle \xi_i(t') \xi_j(t'') \rangle d(t'' - t')
\]

Which substituting in for \( \gamma \) can be re-written:

\[
\eta = \frac{m^2}{36\pi a k_B T} \int_{-\infty}^{\infty} G_{ij}(\tau) d(\tau)
\]

This says that the macroscopic viscosity \( \eta \) is proportional to the correlation function \( G_{ij}(\tau) \) of the microscopic velocity fluctuations. It applies in any system that obeys \( \frac{d\nu}{dt} = -\gamma v + \xi(t) \).

**Markov process**

A markov process is a stochastic process in which the value \( \xi_{n+1} \) depends only upon its immediately prior value \( \xi_n \). We define \( p(a|b) \) \( da \, db \) to be the probability that \( a \) is in the range \((a, a + da)\) given that \( b \) is in the range \((b, b + db)\). For a Markov process this satisfies:

\[
p(\xi_{n+1}|\xi_n, \xi_{n-1} \ldots \xi_0) = p(\xi_{n+1}|\xi_n)
\]

Another useful result is the Chapman-Kolmogorov equation, which states that the probability of \( \xi_2 \) given starting at \( \xi_0 \) is equal to the weighted sum of all possible paths from \( \xi_0 \) to \( \xi_2 \):

\[
p(\xi_2|\xi_0) = \int p(\xi_2|\xi_1) p(\xi_1|\xi_0) \, d\xi_1
\]

A markov process is said to satisfy detailed balance if in the steady state (if it exists) each possible transition is as likely to occur as its time-reversed version.

**Fokker-Planck equation**

The Fokker-Planck equation is a differential equation that describes the time evolution of the probability density function of the velocity of a particle under the influence of drag forces and random forces, as in Brownian motion. The equation is written:

\[
\frac{\partial p(x, t)}{\partial t} = -\frac{\partial}{\partial x_i} \left[ D_i^1(x, t)p(x, t) \right] + \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} \left[ D_{ij}^2(x, t)p(x, t) \right]
\]
Where the $D$s represent the $n$th order cumulants:

$$D^n_{ij} = \lim_{\Delta t \to 0} \left( \frac{\Delta x_i, \Delta x_j \Delta x_k \ldots}{\Delta t} \right)$$

The two coefficients then are:

- Drift coefficient $D^1_{ij}(x, t) = \lim_{\Delta t \to 0} \left( \frac{\Delta x_i}{\Delta t} \right)$, which shifts the peak of the probability distribution to higher $x$ values over time.
- Diffusion coefficient $D^2_{ij}(x, t) = \lim_{\Delta t \to 0} \left( \frac{\Delta x_i, \Delta x_j}{\Delta t} \right)$, which causes the spreading out of the distribution over time.

Every Langevin equation has an associated Fokker-Planck equation. A general Langevin equation of the form:

$$\frac{dx_i}{dt} = A_i(x, t) + B_{ij}(x, t)\xi_j(t)$$

Has a Fokker-Planck equation:

$$\frac{\partial p(x; t)}{\partial t} = -\frac{\partial}{\partial x_i} \left[ A_i(x, t)p(x, t) \right] + \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} \left[ B_{ik}(x, t)B_{jk}(x, t)p(x, t) \right]$$

We seek stationary solutions of the Fokker-Planck equation such that:

$$\frac{\partial p(x; t)}{\partial t} + \frac{\partial}{\partial x_i} \left[ A_i(x, t)p(x, t) \right] - \frac{1}{2} \frac{\partial}{\partial x_j} \left[ B_{ik}(x, t)B_{jk}(x, t)p(x, t) \right] = 0$$

The existence and nature of this solution depends upon the boundary conditions we impose:

- 1. Reflecting barrier: Particle cannot leave an enclosed region, so $n \cdot \mathbf{J}(x, t) = 0$, meaning there is zero probability current through the wall
- 2. Absorbing barrier: Particle removed from system when it touches the boundary. So the probability of being at the boundary vanishes: $p(x, t) = 0$ for $x \in S$
- 3. Discontinuity: Both $A_i$ and $B_{ij}$ can be discontinuous at some surface while still permitting drift and diffusion to occur freely across the surface. Then $J_i(x, t)$ and $p(x, t)$ must be continuous across the surface (i.e. the solutions on both sides must match at $S$), although $\frac{\partial p}{\partial x_i}$ is not necessarily continuous.
- 4. Periodic: An interval $[a, b]$ whose endpoints are identified with each other; usually $A_i$ and $B_{ij}$ are periodic in this situation as well. Then $J_i(x, t)$ and $p(x, t)$ must be periodic.
- 5. Infinity: Normalisation of $p$ usually requires $p \to 0$ and $\frac{\partial p}{\partial x_i} \to 0$ as $x \to \infty$, as long as there is no inflow or outflow of probability current at infinity.

**Backward Fokker-Planck equation**

This equation gives the rate of change in the probability of having come from some initial state changes over time, given a fixed final end state. It has the form:
Where $s$ is some intermediate time $t_0 < s < t$.

With absorbing boundary conditions, the mean exit time for a given particle to leave the system evolves by the backward Fokker-Plank equation:

$$-rac{\partial T(x, t|x', s)}{\partial s} = A_i(x', s) \frac{\partial (x, t|x', s)}{\partial x_i^f} + \frac{1}{2} B_{ij}(x', s) \frac{\partial^2 p(x, t|x', s)}{\partial x_i^f \partial x_j^f}$$

**Einstein-Smoluchowski equation**

This equation is a particular form of the Fokker-Plank equation for describing the spatial diffusion (rather than the velocity diffusion) of Brownian particles. The equation has the form:

$$\frac{\partial q(x, t)}{\partial t} = \gamma \frac{\partial}{\partial x} \left[ \frac{\partial \Phi(x)}{\partial x} q(x, t) \right] + \frac{k_B T}{m \gamma^2} \frac{\partial^2}{\partial x_i \partial x_i} [q(x, t)]$$

It is valid so long as the time taken by the particle to diffuse a distance $l$ satisfies $\frac{m y l^2}{k_B T} \gg \frac{1}{\gamma}$.

**Simple random walk diffusion**

$$\frac{\partial x}{\partial t} = v$$
$$\frac{\partial v}{\partial t} = -\gamma v + \sqrt{D} \xi$$

If $\gamma$ is large we have a strongly damped process, and thus we can approximate $\frac{\partial v}{\partial t} = 0$, hence:

$$v = \frac{1}{\gamma} \sqrt{D} \xi$$
$$\frac{\partial x}{\partial t} = \frac{1}{\gamma} \sqrt{D} \xi$$
$$\frac{\langle \Delta x \rangle}{\Delta t} = \frac{1}{\gamma} \sqrt{D} \xi$$
$$\frac{\langle \Delta x \rangle}{\Delta t} = \frac{1}{\gamma} \sqrt{D} \langle \xi \rangle$$
$$D^1 = 0$$

The $D^1$ term is found by:

$$\langle \Delta x_i \Delta x_i \rangle = \left( \int_0^{\Delta t} \int_0^{\Delta t} \xi(t') \xi(t'') \frac{1}{\gamma} \sqrt{D} \xi(t''') dt' dt'' \right)$$
$$= \left( \int_0^{\Delta t} \int_0^{\Delta t} \xi(t') \xi(t'') \frac{1}{\gamma^2} D dt' dt'' \right)$$
$$= \frac{D}{\gamma^2} \int_0^{\Delta t} \int_0^{\Delta t} \xi(t') \xi(t'') dt' dt''$$
$$= \frac{D}{\gamma^2} \int_0^{\Delta t} \int_0^{\Delta t} \delta(t' - t'') dt' dt''$$
Thus we find the Fokker-Plank equation:

\[
\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x_i} \left[D^i_t(x,t)p(x,t)\right] + \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} \left[D_{ij}(x,t)p(x,t)\right]
\]

\[
\frac{\partial^2 p(x,t)}{\partial x^2} = \frac{D}{2} p(x,t)
\]

**Wiener process**
We begin with the stochastic differential equation:

\[
\frac{dx}{dt} = \xi(t)
\]

Computing the moments we get:

\[
D^1 = \lim_{\Delta t \to 0} \langle \frac{\Delta x}{\Delta t} \rangle = \langle \xi(t) \rangle = 0
\]

\[
D^2 = \lim_{\Delta t \to 0} \left\langle \frac{\Delta x \Delta x}{\Delta t} \right\rangle
\]

\[
= \frac{1}{\Delta t} \int_0^{\Delta t} \int_0^{\Delta t} \xi(t')\xi(t'') dt' dt''
\]

\[
= \frac{1}{\Delta t} \int_0^{\Delta t} \int_0^{\Delta t} \langle \xi(t')\xi(t'') \rangle dt' dt''
\]

\[
= \frac{1}{\Delta t} \int_0^{\Delta t} \int_0^{\Delta t} \delta(t' - t'') dt' dt''
\]

\[
D^2 = 1
\]

Yielding the Fokker-Planck equation:

\[
\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x_i} \left[D^i_t(x,t)p(x,t)\right] + \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} \left[D_{ij}(x,t)p(x,t)\right]
\]

\[
\frac{dp}{dt} = \frac{1}{2} \frac{\partial^2 p}{\partial x^2}
\]

This has the solution:

\[
p(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}
\]
Ornstein-Uhlenbeck process with zero probability current

\[ \frac{dx}{dt} = -\gamma x + D\frac{1}{2}\xi(t) \]

Computing the moments we get:

\[ D^1 = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \left( -\gamma x + D\frac{1}{2}\xi(t) \right) \]

\[ = -\gamma x \]

\[ D^2 = \lim_{\Delta t \to 0} \frac{\Delta x \Delta x}{\Delta t} \]

\[ = \left( \frac{1}{\Delta t} \int_0^{\Delta t} \int_0^{\Delta t} \left( -\gamma x + D\frac{1}{2}\xi(t) \right) \left( -\gamma x + D\frac{1}{2}\xi(t) \right) dt \, dt' \right) \]

\[ = \frac{1}{\Delta t} \int_0^{\Delta t} \int_0^{\Delta t} \left( \gamma^2 x^2 - 2\gamma x D\frac{1}{2}\xi(t) + D\xi(t)\xi(t') \right) dt \, dt' \]

\[ = \frac{1}{\Delta t} \int_0^{\Delta t} \int_0^{\Delta t} \left( \gamma^2 x^2 - D\left( \xi(t)\xi(t') \right) \right) dt \, dt' \]

\[ = \frac{1}{\Delta t} \int_0^{\Delta t} \int_0^{\Delta t} \frac{\Delta t}{\Delta t} \frac{\Delta t}{\Delta t} \left( \gamma^2 x^2 - D\left( \xi(t)\xi(t') \right) \right) dt \, dt' \]

\[ = \frac{1}{\Delta t} \int_0^{\Delta t} \int_0^{\Delta t} \frac{\Delta t}{\Delta t} \left( \gamma^2 x^2 - D\left( \xi(t)\xi(t') \right) \right) dt \, dt' \]

\[ = \frac{1}{\Delta t} \int_0^{\Delta t} \int_0^{\Delta t} \left( \gamma^2 x^2 - D\left( \xi(t)\xi(t') \right) \right) dt \, dt' \]

\[ = \frac{1}{\Delta t} \int_0^{\Delta t} \int_0^{\Delta t} \left( \gamma^2 x^2 - D \right) dt \, dt' \]

\[ = \frac{1}{\Delta t} \int_0^{\Delta t} \int_0^{\Delta t} \left( \gamma^2 x^2 - D \right) dt \, dt' \]

Yielding the Fokker-Planck equation:

\[ \frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x} \left[ D^1(x,t)p(x,t) \right] + \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} \left[ D^2_{ij}(x,t)p(x,t) \right] \]

\[ \frac{dp(x,t)}{dt} = \gamma x \frac{\partial}{\partial x} p(x,t) + D \frac{\partial^2}{\partial x^2} p(x,t) \]

This has the stationary solution:

\[ p_s(x) = \left( \frac{\pi D}{\gamma} \right)^{\frac{1}{2}} \exp \left( -\frac{\gamma x^2}{D} \right) \]

Pendulum in air

We consider a system described by the Langevin equations:

\[ \frac{dx}{dt} = \nu \]

\[ \frac{d\nu}{dt} = -\frac{k}{m} x - \gamma \nu + D^{1/2} \xi(t) \]

To find the Fokker-Planck equation we first need to find all the moments with respect to both \( x \) and \( \nu \).

\[ \frac{\partial p}{\partial t} = -\nu \frac{\partial p}{\partial x} + \frac{k}{m} \frac{\partial p}{\partial \nu} + \gamma \nu \frac{\partial p}{\partial \nu} + D \frac{\partial^2 p}{\partial \nu^2} \]
To find the time evolution of the first moment we consider:

$$\frac{\partial (x)}{\partial t} = \frac{\partial}{\partial t} \int x p(x, v, t) \, dx \, dv$$

$$= \int \int x \frac{\partial p}{\partial t} \, dx \, dv$$

$$= \int \int -v x \frac{\partial p}{\partial x} + \frac{k}{m} x^2 \frac{\partial p}{\partial v} + \gamma p x + \gamma v x \frac{\partial p}{\partial v} + D x \frac{\partial^2 p}{\partial v^2} \, dx \, dv$$

$$\frac{\partial (x)}{\partial t} = -\int \int v x \frac{\partial p}{\partial x} \, dx \, dv + \frac{k}{m} \int \int x^2 \frac{\partial p}{\partial v} \, dx \, dv + \gamma \int \int p x \, dx \, dv + \gamma \int x \frac{\partial p}{\partial v} \, dx \, dv + D \int \int x \frac{\partial^2 p}{\partial v^2} \, dx \, dv$$

To solve these integrals we need to use integration by parts and the boundary condition that probabilities should go to zero at infinity. Thus we have:

$$\int \int v x \frac{\partial p}{\partial x} \, dx \, dv = v x p \big|_{-\infty}^{\infty} - \int \int v p \, dx \, dv = -\langle v \rangle$$

$$\int \int x^2 \frac{\partial p}{\partial v} \, dx \, dv = x^2 p \big|_{-\infty}^{\infty} - \int \int p x \, dx \, dv = 0$$

$$\int \int p x \, dx \, dv = \langle x \rangle$$

$$\int \int v x \frac{\partial p}{\partial v} \, dx \, dv = v x p \big|_{-\infty}^{\infty} - \int \int x p \, dx \, dv = -\langle x \rangle$$

$$\int \int x \frac{\partial^2 p}{\partial v^2} \, dx \, dv = x \frac{\partial p}{\partial v} \big|_{-\infty}^{\infty} - \int \int \frac{\partial p}{\partial v} \, dx \, dv = 0$$

Putting the pieces together we get:

$$\frac{\partial (x)}{\partial t} = \langle v \rangle + 0 + \gamma \langle x \rangle - \gamma \langle x \rangle + 0$$

$$\frac{\partial (x)}{\partial t} = \langle v \rangle$$

Now for the time evolution of the second moment of $x$:

$$\frac{\partial (x^2)}{\partial t} = -\int \int v x^2 \frac{\partial p}{\partial x} \, dx \, dv + \frac{k}{m} \int \int x^3 \frac{\partial p}{\partial v} \, dx \, dv + \gamma \int \int p x^2 \, dx \, dv + \gamma \int \int v x^2 \frac{\partial p}{\partial v} \, dx \, dv$$

$$+ D \int \int x^2 \frac{\partial^2 p}{\partial v^2} \, dx \, dv$$

$$\int \int v x^2 \frac{\partial p}{\partial x} \, dx \, dv = v x^2 p \big|_{-\infty}^{\infty} - 2 \int \int v x p \, dx \, dv = -2 \langle x v \rangle$$

$$\int \int x^3 \frac{\partial p}{\partial v} \, dx \, dv = x^3 p \big|_{-\infty}^{\infty} - \int \int p \, dx \, dv = 0$$

$$\int \int p x^2 \, dx \, dv = \langle x^2 \rangle$$

$$\int \int v x^2 \frac{\partial p}{\partial v} \, dx \, dv = v x^2 p \big|_{-\infty}^{\infty} - \int \int x^2 p \, dx \, dv = -\langle x^2 \rangle$$

$$\int \int x^2 \frac{\partial^2 p}{\partial v^2} \, dx \, dv = x^2 \frac{\partial p}{\partial v} \big|_{-\infty}^{\infty} - \int \int \frac{\partial p}{\partial v} \, dx \, dv = 0$$
Putting the pieces together we get:

\[
\frac{\partial (x^2)}{\partial t} = 2(xv)
\]

Now for the time evolution of the second moment of \(x\):

\[
\frac{\partial (v^2)}{\partial t} = - \int \int v^3 \frac{\partial p}{\partial x} \ dx \ dv + \frac{k}{m} \int \int xv^2 \frac{\partial p}{\partial v} \ dx \ dv + \gamma \int \int pv^2 \ dx \ dv + \gamma \int \int v^3 \frac{\partial p}{\partial v} \ dx \ dv
\]

\[
+ \frac{D}{2} \int \int v^2 \frac{\partial^2 p}{\partial v^2} \ dx \ dv
\]

\[
\int v^3 \frac{\partial p}{\partial x} \ dx \ dv = v^3 p\big|_{-\infty}^{\infty} - \int v \times 0 \ dx \ dv = 0
\]

\[
\int xv^2 \frac{\partial p}{\partial v} \ dx \ dv = xv^2 p\big|_{-\infty}^{\infty} - 2 \int xv \ dx \ dv = -2(xv)
\]

\[
\int pv^2 \ dx \ dv = \langle v^2 \rangle
\]

\[
\int v^3 \frac{\partial p}{\partial v} \ dx \ dv = v^3 p\big|_{-\infty}^{\infty} - 3 \int v^2 p \ dx \ dv = -3\langle v^2 \rangle
\]

\[
\int v^2 \frac{\partial^2 p}{\partial v^2} \ dx \ dv = v^2 \frac{\partial p}{\partial v}\big|_{-\infty}^{\infty} - 2 \int \frac{\partial p}{\partial v} v \ dx \ dv = 2
\]

Putting the pieces together we get:

\[
\frac{\partial (v^2)}{\partial t} = -2 \frac{k}{m} \langle xv \rangle - 2\gamma \langle v^2 \rangle + D
\]

Now for the time evolution of the mixed moment:

\[
\frac{\partial (xv)}{\partial t} = - \int \int v^2 x \frac{\partial p}{\partial x} \ dx \ dv + \frac{k}{m} \int \int x^2 v \frac{\partial p}{\partial v} \ dx \ dv + \gamma \int \int pxv \ dx \ dv + \gamma \int \int v^2 x \frac{\partial p}{\partial v} \ dx \ dv
\]

\[
+ \frac{D}{2} \int \int xv \frac{\partial^2 p}{\partial v^2} \ dx \ dv
\]

\[
\int v^2 x \frac{\partial p}{\partial x} \ dx \ dv = v^2 xp\big|_{-\infty}^{\infty} - \int v^2 p \ dx \ dv = -\langle v^2 \rangle
\]

\[
\int x^2 v \frac{\partial p}{\partial v} \ dx \ dv = x^2 vp\big|_{-\infty}^{\infty} - \int px^2 \ dx \ dv = -\langle x^2 \rangle
\]

\[
\int pxv \ dx \ dv = \langle xv \rangle
\]

\[
\int v^2 x \frac{\partial p}{\partial v} \ dx \ dv = v^2 xp\big|_{-\infty}^{\infty} - 2 \int vxp \ dx \ dv = -2\langle xv \rangle
\]

\[
\int xv \frac{\partial^2 p}{\partial v^2} \ dx \ dv = xv \frac{\partial p}{\partial v}\big|_{-\infty}^{\infty} - \int \frac{\partial p}{\partial v} x \ dx \ dv = 0
\]

Putting the pieces together we get:

\[
\frac{\partial (xv)}{\partial t} = \langle v^2 \rangle - \frac{k}{m} \langle x^2 \rangle - \gamma \langle xv \rangle
\]
Birth and Death Processes

Introduction

Birth-death processes involve populations of discrete numbers of individuals that make transitions between states. Individuals in particular states can be created or destroyed via creation and destruction operators.

A simple example is a predator-prey ecology which can be described using the relations:

\[ \begin{align*}
X + A &\xrightarrow{k_1} 2X \quad \text{prey eats and reproduces} \\
X + Y &\xrightarrow{k_2} 2Y \quad \text{predator eats and reproduces} \\
Y &\xrightarrow{k_3} 0 \quad \text{predator dies}
\end{align*} \]

If populations are large we can approximate the population densities by continuous variables, where \( x \) for instance is the number of prey per unit of land area. We then get a system of ODEs known as the Lotka-Volterra equations:

\[
\begin{align*}
\frac{dx}{dt} &= k_1 ax - k_2 xy \\
\frac{dy}{dt} &= k_2 xy - k_3 y
\end{align*}
\]

Carrying capacity can be incorporated into this model by incorporating a negative nonlinear term:

\[
\frac{dx}{dt} = k_1 x k - k_2 x^2
\]

This has a solution called the logistic equation, which gives a sigmoid curve:

\[
x(t) = \frac{x_0 \exp(k_1 t)}{1 + \frac{k_2}{k_1} x_0 (\exp(k_1 t) - 1)}
\]

The discrete version of the logistic equation, called a logistic map, has chaotic solutions:

\[
x_{n+1} = k_1 x_n - k_2 x_n^2
\]

Introducing time-delays into the equations has the effect of introducing cyclic behaviour in the solutions, for example:

\[
\frac{dx(t)}{dt} = -k x(t - T)
\]

Master equations

In more realistic birth-death processes, populations fluctuate stochastically about deterministic trends. This can be modelled by replacing the continuous ODEs by a master equation describing discrete jumps between different possible states.

For the Lotka-Volterra ecology the possible processes are:

\[
p(x \rightarrow x + 1, y \rightarrow y) = k_1 ax \Delta t \\
p(x \rightarrow x - 1, y \rightarrow y + 1) = k_2 xy \Delta t
\]
This leads to a probability distribution what is the sum of all possible transition processes weighted by their probabilities of occurring:

$$
p(x \rightarrow x, y \rightarrow y - 1) = k_3 y \Delta t 
\quad p(x \rightarrow x, y \rightarrow y) = 1 - (k_1 x + k_2 xy + k_3 y) \Delta t
$$

Resulting in a master equation of the form:

$$
\frac{\partial p(x, y, t + \Delta t)}{\partial t} = k_1 a(x - 1)p(x - 1, y, t)\Delta t + k_2 (x + 1)(y - 1)p(x + 1, y - 1, t)\Delta t
\quad + k_3 (y + 1)p(x, y + 1, t)\Delta t + [1 - (k_1 x + k_2 xy + k_3 y) \Delta t]p(x, y, t)
$$

In general terms, a master equation is a Chapman-Kolmogorov equation for Markov processes. It governs the evolution of state occupation numbers by summing up the gain and loss probabilities into and out of each state. The easiest way to determine a master equation is to sum all the probability-weighted ways of getting to the desired final state, and then subtract all the probability-weighted ways of leaving that state.

**Two state quantum system**

Consider a two level quantum system with energies $E_1$ and $E_2$ and transition rates $t_{12}$ and $t_{21}$. We first note that since there is only one particle we have:

$$
p(E_1, t) + p(E_2, t) = 1
\quad p(E_2, t) = 1 - p(E_1, t)
$$

The master equation for this system is then given by:

$$
p(E_1, t + \Delta t) = t_{21} p(E_2, t) \Delta t - (1 - t_{12})p(E_1, t) \Delta t
\quad = t_{21} (1 - p(E_1, t)) \Delta t - (1 - t_{12})p(E_1, t) \Delta t
\quad = (t_{21} - t_{21} p(E_1, t) - p(E_1, t) + t_{12} p(E_1, t)) \Delta t
\quad = (t_{21} - p(E_1, t)(t_{21} + t_{12} - 1)) \Delta t
\quad \frac{\partial p(E_1, t)}{\partial t} = t_{21} - (t_{21} + t_{12} - 1)p(E_1, t)
$$

First solve the homogenous version of this equation:

$$
\frac{\partial p(E_1, t)}{\partial t} = -(t_{21} + t_{12} - 1)p(E_1, t)
\quad p_h(E_1, t) = A_0 e^{-(t_{21} + t_{12} - 1)t}
$$

Write the general solution as:

$$
p(E_1, t) = k_1 + A_0 e^{-(t_{21} + t_{12} - 1)t}
$$

Substituting in:

$$
-(t_{21} + t_{12} - 1)e^{-(t_{21} + t_{12} - 1)t} = t_{21} - (t_{21} + t_{12} - 1)(k_1 + e^{-(t_{21} + t_{12} - 1)t})
\quad -(t_{21} + t_{12} - 1)e^{-(t_{21} + t_{12} - 1)t} = t_{21} - ((t_{21} + t_{12} - 1)k_1 + (t_{21} + t_{12} - 1)e^{-(t_{21} + t_{12} - 1)t})
\quad -(t_{21} + t_{12} - 1)e^{-(t_{21} + t_{12} - 1)t} = t_{21} - (t_{21} + t_{12} - 1)k_1 - (t_{21} + t_{12} - 1)e^{-(t_{21} + t_{12} - 1)t}
\quad t_{21} = (t_{21} + t_{12} - 1)k_1
$$

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\[ k_1 = \frac{t_{21}}{t_{21} + t_{12} - 1} \]

Which leaves us with the solution:

\[ p(E_1, t) = \frac{t_{21}}{t_{21} + t_{12} - 1} + A_0 e^{-(t_{21} + t_{12} - 1) t} \]

Now we can compute the expected value of \( E_1 \) in the limit \( t \to \infty \):

\[ \langle E \rangle = p(E_1, t) E_1 + p(E_2, t) E_2 \]
\[ = \frac{t_{21} E_1}{t_{21} + t_{12} - 1} + \frac{(1 - t_{21}) E_2}{t_{21} + t_{12} - 1} \]
\[ = \frac{E_2 - t_{21} E_2 + t_{21} E_1}{t_{21} + t_{12} - 1} \]

**Bimolecular chemical reaction**

Consider the following chemical reaction:

\[ X \xrightleftharpoons[k_2]{k_1} A \]

Which has species concentration \( x \) and \( a \). The resulting master equation is:

\[ p(x, t + \Delta t) = k_2 a p(x - 1, t) \Delta t + k_1 (x + 1) p(x + 1, t) \Delta t - k_1 x p(x, t) \Delta t - k_2 a p(x, t) \]
\[ \frac{\partial p(x, t)}{\partial t} = k_2 a p(x - 1, t) + k_1 (x + 1) p(x + 1, t) - (k_1 x + k_2 a) p(x, t) \]

We solve this using the generating function:

\[ G(s, t) = \sum_{x=0}^{\infty} s^x p(x, t) \]

Substituting this into the master equation we get (using \( p(-1, t) = 0 \)):

\[ \frac{\partial G(s, t)}{\partial t} = \sum_{x=0}^{\infty} s^x \frac{\partial p(x, t)}{\partial t} \]
\[ = k_2 a \sum_{x=0}^{\infty} s^x p(x - 1, t) + k_1 \sum_{x=0}^{\infty} s^x (x + 1) p(x + 1, t) - k_1 \sum_{x=0}^{\infty} s^x x p(x, t) - k_2 a \sum_{x=0}^{\infty} s^x p(x, t) \]
\[ = k_2 a \sum_{x'=1}^{\infty} s^{x'-1} p(x', t') + k_1 \sum_{x'=1}^{\infty} s^{x'-1} (x') p(x', t) - k_1 \sum_{x=0}^{\infty} s^x x p(x, t) - k_2 a \sum_{x=0}^{\infty} s^x p(x, t) \]
\[ = k_2 a s \sum_{x=0}^{\infty} s^x p(x, t) - k_2 a \sum_{x=0}^{\infty} s^x p(x, t) + k_1 \sum_{x=0}^{\infty} s^{x-1} x p(x, t) - k_1 \sum_{x=0}^{\infty} s^x x p(x, t) \]
\[ = k_2 a (s - 1) \sum_{x=0}^{\infty} s^x p(x', t') + k_1 (1 - s) \sum_{x=0}^{\infty} s^{x-1} (x') p(x', t) \]
\[ \frac{\partial G(s, t)}{\partial t} = k_2 a (s - 1) G(s, t) + k_1 (1 - s) \frac{\partial G(s, t)}{\partial s} \]

We can use this to compute moments of \( p(x, t) \):
The BBGKY hierarchy is a set of equations describing the dynamics of a system of a large number of interacting particles. It begins with the Liouville equation, which describes the full microscopic distribution function $f_N(t, x, p) d^3N x d^3N p$ under the action of the Hamiltonian $H$:

\[
\frac{\partial f_N}{\partial t} + \frac{\partial H}{\partial p_i} \frac{\partial f_N}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial f_N}{\partial p_i} = 0
\]

The BBGKY hierarchy is a reformulation of this equation which writes $f_1$ in terms of $f_2$, and $f_2$ in terms of $f_3$, and so on, where $f_n$ incorporates only $n + 1$ body interactions. How many orders when needed to include to get a good approximation of the system depends on the density of the system.

The $s$ particle distribution function evolves as:

\[
\frac{\partial f_s}{\partial t} = -\hbar_s f_s - \sum_{i=1}^{s} \int \frac{\partial f_{s+1}}{\partial p_i} d^3x_{s+1} d^3p_{s+1}
\]

The Boltzmann equation involves only the two lowest-order BBGKY equations, $s = 1, 2$. For a dilute gas this becomes for the first order term:

\[
\left( \frac{\partial}{\partial t} + \frac{p_1}{m} \frac{\partial}{\partial x_1} + F_1 \frac{\partial}{\partial p_1} \right) f_1(t, x_1, p_1) = -\int_{|x_1-x_2|<r_0} K_{12} \frac{\partial}{\partial p_1} f_2(t, x_1, x_2, p_1, p_2) d^3x_2 d^3p_2
\]

And for the second order term:

\[
\left( \frac{\partial}{\partial t} + \frac{p_1}{m} \frac{\partial}{\partial x_1} + \frac{p_2}{m} \frac{\partial}{\partial x_2} + F_1 \frac{\partial}{\partial p_1} + F_2 \frac{\partial}{\partial p_2} + \frac{1}{2} K_{12} \left( \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right) \right) f_2(t, x_1, x_2, p_1, p_2) = 0
\]

Where $F_i = \frac{\partial U}{\partial x_i}$ and $K_{ij} = \frac{\partial^2 V(|x_i-x_j|)}{\partial x_i \partial x_j}$, given a Hamiltonian of the form:

\[
H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i=1}^{N} U(x_i) + \sum_{i=1}^{N} \sum_{j>i}^{N} V(|x_i-x_j|)
\]

If the effect of collisions is to restore a local equilibrium of $f^0(x, v, t)$ with characteristic relaxation time $\tau$, then in the linear response limit the Boltzmann equation takes a special form known as the BGK or relaxation approximation:

\[
\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} + \frac{F}{m} \frac{\partial}{\partial v} \right) f(t, x_1, p_1) = -\frac{f - f^0}{\tau}
\]
An even stronger simplification ignores the effect of interactions altogether, and has the form:

\[
\left(\frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial x} + \frac{E}{m} \cdot \frac{\partial}{\partial v}\right) f(t, x, p) = 0
\]

**Electrical conductivity**

Consider a small electric field \( E^1 \) applies to a uniform gas containing ions of charge \( q \) and mass \( m \). This field will cause a departure from equilibrium in the form of an electrical current \( f^1 = \sigma_{ij} E_j^1 \), where \( \sigma_{ij} \) is the electrical conductivity. We consider a perturbation of the form \( f = f^0 + f^1 \) (where \( f \) is the particle density distribution), and use the relaxation form of the Boltzmann equation:

\[
\frac{\partial f^1}{\partial t} + v \cdot \frac{\partial f^1}{\partial x} + \frac{qE^1}{m} \cdot \frac{\partial}{\partial v} (f^0 + f^1) = -\frac{f^1}{\tau}
\]

Since we are considering a small perturbation the second-order term \( E^1 \cdot \frac{\partial f^1}{\partial v} \approx 0 \). We will also assume that \( E^1 \) is static and uniform, meaning \( f^1 \) is also static and uniform and hence \( \frac{\partial f^1}{\partial x} = \frac{\partial f^1}{\partial t} = 0 \). Hence:

\[
\frac{qE^1}{m} \cdot \frac{\partial f^0}{\partial v} = -\frac{f^1}{\tau}
\]

\[
f^1 = -\tau \frac{qE^1}{m} \cdot \frac{\partial f^0}{\partial v}
\]

By definition the current density is equal to charge times average velocity:

\[
f^1 = \int v f^1 q \, d^3v
\]

\[
f^1 = -\frac{q^2 \tau E^1}{m} \int v \cdot \frac{\partial f^0}{\partial v} \, d^3v
\]

If we take the field to be along the x-axis this simplifies to:

\[
J_x^1 = -\frac{q^2 \tau E^1}{m} \int v_x f^0 \, d^3v
\]

\[
= -\frac{q^2 \tau E^1}{m} \left( \left. v_x f^0 \right|_\infty - \int f^0 \, d^3v \right)
\]

\[
= \frac{q^2 \tau E^1}{m} \int f^0 \, d^3v
\]

\[
J_x^1 = \frac{q^2 \tau n_0 E^1}{m}
\]

Since the final integral is just the number of particles per unit volume. Thus we have:

\[
J_x^1 = \frac{q^2 \tau n_0 E^1}{m}
\]

Thereby giving an expression for conductivity:

\[
\sigma_x = \frac{q^2 \tau n_0}{m}
\]
**Transport**

In the dilute limit, the rate at which material is transported is given by:

\[
\frac{df}{dt} + v_1 \frac{\partial f}{\partial x_1} + \frac{F_1}{m} \frac{\partial f}{\partial v_1} = -\int \delta(v_1 - v_2)(f(v_1)f(v_2) - f(v_1)f(v_2)) \, dv_1 dv_2 dv_2
\]

**Stability of Non-Equilibrium Systems**

**Linear stability analysis**

After we have found a fixed point for a system of nonlinear ODEs, we often want to know whether this point is stable to small perturbations. To examine the stability of \(x(0)\), it is enough to linearise about it and solve for the fate of infinitesimal disturbances, as the nonlinear dynamics are irrelevant to local (as opposed to global) stability in all situations where linearization does indeed lead to a linear system. We explore this through a worked example.

We begin with the system of equations:

\[
\begin{align*}
\frac{dx}{dt} &= ax - \beta xy \\
\frac{dy}{dt} &= \beta yx - y
\end{align*}
\]

Equilibrium occurs when:

\[
\begin{align*}
0 &= ax - \beta xy \\
0 &= \beta yx - y
\end{align*}
\]

Solving we find:

\[
\begin{align*}
x &= 0, \ y = 0 \\
x &= \frac{1}{\beta}, \ y = \frac{a}{\beta}
\end{align*}
\]

We perform a linear stability analysis about each fixed point separately.

For \(x = y = 0\):

\[
\begin{align*}
x &= x_0 + x_1(t) = x_1(t) \\
y &= y_0 + y_1(t) = y_1(t)
\end{align*}
\]

We linearise the system about the fixed point:

\[
\begin{align*}
\frac{d}{dt} (x) &\approx \frac{d}{dt} (x_0) + J_{(x_0,y_0)} (x_1, y_1) \\
\frac{d}{dt} (y) &\approx J_{(x_0,y_0)} (x_1, y_1)
\end{align*}
\]

Now find the Jacobian matrix:

\[
J_{(x_0,y_0)} = \begin{bmatrix}
\frac{\partial x_1}{\partial x} & \frac{\partial x_1}{\partial y} \\
\frac{\partial y_1}{\partial x} & \frac{\partial y_1}{\partial y}
\end{bmatrix}_{(x_0,y_0)}
\]
Hence we have:

\[
\frac{d}{dt} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}
\]

Which has solutions:

\[
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = c_1 \tilde{v}_1 e^{\alpha t} + c_2 \tilde{v}_2 e^{-t}
\]

One of these eigenvalues values \( \alpha > 0 \), so this equilibrium state is unstable.

Now for \( x = \frac{1}{\beta}, y = \frac{\alpha}{\beta} \):

\[
x = x_0 + x_1(t) = \frac{1}{\beta} + x_1(t)
\]
\[
y = y_0 + y_1(t) = \frac{\alpha}{\beta} + y_1(t)
\]

We linearise the system about the fixed point:

\[
\frac{d}{dt} (x) \approx \frac{d}{dt} (x_0) + J_{(x_0,y_0)} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}
\]

Now find the Jacobian matrix:

\[
J_{(x_0,y_0)} = \begin{bmatrix} \frac{\partial x_1}{\partial x} & \frac{\partial x_1}{\partial y} \\ \frac{\partial y_1}{\partial x} & \frac{\partial y_1}{\partial y} \end{bmatrix}_{(x_0,y_0)}
\]

\[
= \begin{bmatrix} \alpha - \beta y & -\beta x \\ \beta y & \beta x - 1 \end{bmatrix}_{(x_0,y_0)}
\]

\[
= \begin{bmatrix} 0 & -1 \\ \alpha & 0 \end{bmatrix}
\]

Hence we have:

\[
\frac{d}{dt} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ \alpha & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}
\]

The eigenvalues of this matrix are \( \lambda = \pm i \sqrt{\alpha} \), which is spiral behaviour.

Generally, the possibilities for singularities are as follows:

- \( \lambda_1 \) and \( \lambda_2 \) are distinct and real with the same sign: forms a stable node if the eigenvalues are negative and an unstable node if they are positive.
- \( \lambda_1 \) and \( \lambda_2 \) are distinct and real with different signs: the behaviour depends upon the direction of approach.
- $\lambda_1$ and $\lambda_2$ are complex with no real component: phase curves are ellipses with a singularity at the centre.
- $\lambda_1$ and $\lambda_2$ are complex with a real component: has spiral singularities which are stable if the real component is negative, and unstable if it is positive.

**Limit cycles**

A Hopf bifurcation occurs when a fixed point in a two-dimensional system controlled by a single parameter $\nu$ makes a transition from a stable spiral, to a limit cycle growing out of the origin. To see how to determine where such bifurcations occur, consider the following example.

We have the non-linear system:

$$\frac{dx}{dt} = Ax + By \quad p = A + D$$
$$\frac{dy}{dt} = Cx + Dy \quad q = AD - BC$$
$$\Delta = p^2 - 4q$$

The equilibrium point is $x_1 = x_2 = 0$. Now we find the Jacobian:

$$J_{(x_0,y_0)} = \begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial y_1} \\ \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial y_1} \end{bmatrix}_{(x_0,y_0)}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 - 2x_1 & \nu \end{bmatrix}_{(x_0,y_0)}$$

The eigenvalues of this matrix are:
We want to consider the cutoff point when these are purely imaginary. Here this occurs with \( \nu = 0 \), when \( \lambda = \pm i \). To determine if there is a limit cycle, we need to consider the matrix:

\[
B(\nu) = \frac{I(\nu) - I(0)}{\nu} = \frac{1}{\nu} \begin{bmatrix} 0 & 1 \\ -1 & \nu \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

If \( \text{Tr}(B) \neq 0 \) then the original system has a periodic solution for \( \nu \) with period \( \approx 2\pi/|\text{Im}(\lambda(\nu = 0))| \). In this case the trace is clearly nonzero, in which case the original system has a periodic solution with period \( 2\pi \), and a Hopf bifurcation occurring at \( \nu = 0 \).

**Time-dependent coefficients**

In general we may have a set of linear equations where the coefficients are time-dependent:

\[
\frac{dx_i(t)}{dt} = A_{ij}(t)x_j(t)
\]

In an \( n \)-dimensional such system, there will be \( n \) linearly independent and complete solutions \( u^1(t), \ldots, u^n(t) \). We define \( \Phi_{ij} = [\tilde{u}^1, \tilde{u}^2, \ldots, \tilde{u}^n] \) to be the matrix whose columns are these independent solutions. This is called the fundamental matrix.

The solution to the above set of equations is given by:

\[
x(t) = \Phi(t)\Phi^{-1}(t_0)x_0
\]

If there is an additional forcing term \( F(t) \), the full solution takes the form:

\[
x(t) = \Phi(t)\Phi^{-1}(t_0)x(t_0) + \Phi(t_0) \int_{t_0}^{t} \Phi^{-1}(t')F(t') \, dt'
\]

**Resonance**

An open system with time-dependent parameters will, if undamped, typically exhibit an infinite response when the parameters oscillate at a simple fraction multiple of the natural frequency of the system. To illustrate this we consider the following example. Begin with the equation:

\[
\frac{d^2 x}{dt^2} + \omega_0(1 + \epsilon \cos \omega t)x = 0
\]

The strongest response will occur at \( \omega = 2\omega_0 \), so set \( \omega = 2\omega_0 + \sigma \) for a small perturbation \( \sigma \). We then look for solutions of the form:

\[
x(t) = A(\sigma t) \cos \omega t + B(\sigma t) \sin \omega t
\]
The coefficients are slowly varying parameters of time, hence $\sigma t$. We can solve for the exact form of these coefficients by substituting back into the original equation and ignoring all terms that do not resonate at $\omega_0$:

$$
2A' + B\sigma + \frac{1}{2} \epsilon \omega_0 B = 0
$$

$$
2B' - A\sigma + \frac{1}{2} \epsilon \omega_0 A = 0
$$

To find solutions we write this in matrix form:

$$
\frac{d}{dt} \begin{pmatrix} A' \\ B' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & \sigma + \frac{1}{2} \epsilon \omega_0 \\ -\sigma + \frac{1}{2} \epsilon \omega_0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}
$$

We are looking for resonating solutions, so we want solutions that grow with time, and thus have $\lambda > 0$. Solving for the eigenvalues we find this condition is satisfied for:

$$
|\lambda| \geq \frac{1}{16} \sqrt{\frac{\epsilon^2 \omega_0^2}{4} - \sigma^2}
$$

Which is satisfied for perturbations:

$$
-\frac{\epsilon \omega_0}{2} \leq \sigma \leq \frac{\epsilon \omega_0}{2}
$$

**Forced nonlinear harmonic oscillators**

In a non-linear oscillator, the output will not be a simple superposition of the initial input waves. If we consider the case of a quadratic nonlinearity we have instead:

$$
V_{out}(t) = k[V_{in}(t) + e V_{in}(t)^2]
$$

We consider the case of input consisting of two pure tones $V_{in}(t) = A \cos \omega_1 t + B \cos \omega_2 t$:

$$
V_{out}(t) = k(A \cos \omega_1 t + B \cos \omega_2 t) + k\epsilon (A \cos \omega_1 t + B \cos \omega_2 t)^2
$$

$$
V_{out}(t) = \frac{1}{2} k\epsilon (A^2 + B^2) + kA \cos \omega_1 t + \frac{1}{2} k\epsilon A^2 \cos 2\omega_1 t + kB \cos \omega_2 t + \frac{1}{2} k\epsilon B^2 \cos 2\omega_2 t + k\epsilon AB \cos (\omega_1 + \omega_2) t + k\epsilon AB \cos (\omega_1 - \omega_2) t
$$

Note that beat notes enter the output as a result of the sum and difference frequencies $\omega_1 \pm \omega_2$.

A simple harmonic oscillator is linear:

$$
\frac{d^2x}{dt^2} + \omega_0^2 x = 0
$$

Adding a damping term and a driving factor causes a phase shift so that output leads or lags the driving force, and in general also changes the amplitude, in this case by the relation $A = \frac{F_0}{\omega_0^2 - \omega^2 - \omega \gamma}$.

$$
\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = F_0 \cos \omega t
$$
A nonlinear harmonic oscillator has an $x$ term with power greater than 1. Stable nonlinear oscillations result in a period-amplitude relation, in this case of the form $\omega = \omega_0 + \frac{3\epsilon}{8\omega_0} |A|^2$.

$$\frac{d^2x}{dt^2} + \epsilon x^3 + \omega_0^2 x = 0$$

A damped, driven, nonlinear harmonic oscillator has all three of the above terms, and can exhibit very complex behaviour.

$$\frac{d^2x}{dt^2} + \epsilon x^3 + \gamma \frac{dx}{dt} + \omega_0^2 x = F_0 \cos \omega t$$

Combining both forcing and nonlinear effects into a single equation we get the period-amplitude relation:

$$|A|^2 = \frac{F_0^2}{\gamma^2 \omega_0^2 + \frac{3\epsilon \gamma^2}{4} |A|^2 + \frac{9\epsilon^2}{16} \left(1 + \frac{\gamma^2}{4\omega_0^2}\right) |A|^4}$$

This system exhibits bistability and hysteresis.

Pattern Formation

Pattern formation is the study of periodic spatial variations in the solutions of nonlinear differential equations. We typically modern the process of pattern formation as occurring by perturbations which spread as waves throughout the system.

We begin with the coupled reaction-diffusion equations:

$$\frac{\partial n}{\partial t} = D \nabla^2 n - a \nabla \cdot (n \nabla c) + s r n (N - n)$$
$$\frac{\partial c}{\partial t} = \nabla^2 c + \frac{s n}{1 + n} - s c$$

To find the equilibrium we set both derivatives to zero:

$$0 = D \nabla^2 n - a \nabla \cdot (n \nabla c) + s r n (N - n)$$
$$0 = \nabla^2 c + \frac{s n}{1 + n} - s c$$

The equilibrium is specified to be uniform, so spatial derivatives are also set to zero:

$$0 = s r n (N - n)$$
Yielding:

\[ n_0 = N \]
\[ c_0 = \frac{N}{1 + N} \]

Now we ‘switch off’ diffusion, which means we set all spatial derivatives to zero:

\[ \frac{\partial n}{\partial t} = sr(n - n) \]
\[ \frac{\partial c}{\partial t} = sn - sc \]

We then conduct a linear stability analysis about the uniform equilibrium \((n_0, c_0)\):

\[
J_{(n_0, c_0)} = \begin{bmatrix}
\frac{\partial \dot{n}}{\partial n} & \frac{\partial \dot{n}}{\partial c} \\
\frac{\partial \dot{c}}{\partial n} & \frac{\partial \dot{c}}{\partial c}
\end{bmatrix}
\]

\[ = \begin{bmatrix}
\frac{srN - 2sr}{s} & 0 \\
\frac{sn}{1 + n} - \frac{(1 + n)^2}{sn} & -s
\end{bmatrix}_{(n_0, c_0)} \]

\[ = \begin{bmatrix}
\frac{srN - 2sr}{s} & 0 \\
\frac{sN}{1 + N} - \frac{(1 + N)^2}{sN} & -s
\end{bmatrix} \]

\[ \det(J - \lambda I) = (srN - 2sr - \lambda)(-s - \lambda) \]
\[ 0 = (srN - 2sr - \lambda)(s + \lambda) \]
\[ 0 = (s^2rN - 2s^2r - \lambda s + (srN - 2sr)\lambda - \lambda^2) \]
\[ 0 = (s^2rN - 2s^2r) + (srN - 2sr - s)\lambda - \lambda^2 \]

We now linearise the equation about the point \((n_0, c_0)\) by demanding a solution of the form:

\[ n = n_0 + n_1 \]
\[ c = c_0 + c_1 \]

Where:

\[ \frac{\partial n_1}{\partial t} = f_{11}(n_0, c_0)n_1 + f_{12}(n_0, c_0)c_1 \]
\[ \frac{\partial c_1}{\partial t} = f_{12}(n_0, c_0)n_1 + f_{22}(n_0, c_0)c_1 \]

Substituting this into our original equations we get:

\[ \frac{\partial (n_0 + n_1)}{\partial t} = D\nabla^2(n_0 + n_1) - a\nabla \cdot ((n_0 + n_1)\nabla(c_0 + c_1)) + sr(n_0 + n_1)(N - (n_0 + n_1)) \]
\[ \frac{\partial (c_0 + c_1)}{\partial t} = \nabla^2(c_0 + c_1) + \frac{s(n_0 + n_1)}{1 + (n_0 + n_1)} - s(c_0 + c_1) \]
Since $c_0$ and $n_0$ are spatially uniform equilibrium points, and also substituting in the equilibrium itself, we can simplify this as:

\[
\frac{\partial n_1}{\partial t} = D\nabla^2 n_1 - a\nabla \cdot (N\nabla c_1 + n_1\nabla c_1) + sr(N + n_1)(-n_1) \\
\frac{\partial c_1}{\partial t} = \nabla^2 c_1 + \frac{s(N + n_1)}{1 + N + n_1} - s \left( \frac{N}{1 + N} + c_1 \right)
\]

Comparing to the original forms of the derivatives given above, we note that we do not need to keep any $n_1^2$, $c_1^2$, or $n_1c_1$ terms, or any constant terms, so discarding these we are left with:

\[
\frac{\partial n_1}{\partial t} = D\nabla^2 n_1 - aN\nabla^2 c_1 - srNn_1 \\
\frac{\partial c_1}{\partial t} = \nabla^2 c_1 + \frac{s(N + n_1)}{1 + N + n_1} - sc_1
\]

We also need to make the approximation using $N \gg n_1$:

\[
s\left( \frac{N + n_1}{1 + N} \right) = \frac{1}{1 + N} \frac{s(N + n_1)}{\left( 1 + \frac{n_1}{1 + N} \right)} \\
\approx \frac{1}{1 + N} s(N + n_1) \left( 1 - \frac{n_1}{1 + N} \right)
\]

Thus we finally arrive at our linearised equations:

\[
\frac{\partial n_1}{\partial t} = D\nabla^2 n_1 - srNn_1 - aN\nabla^2 c_1 \\
\frac{\partial c_1}{\partial t} = \nabla^2 c_1 - sc_1 + \frac{s}{(1 + N)^2} n_1
\]

We now consider plane wave perturbations of the form:

\[n_1 = Ae^{(ik\cdot\bar{x} + \lambda t)}, \quad c_1 = Be^{(ik\cdot\bar{x} + \lambda t)}\]

Substituting these into the RHS we have (watch for minus signs!):

\[
\frac{\partial n_1}{\partial t} = -Dk^2Ae^{(ik\cdot\bar{x} + \lambda t)} - srNAe^{(ik\cdot\bar{x} + \lambda t)} + aNk^2Be^{(ik\cdot\bar{x} + \lambda t)} \\
\frac{\partial c_1}{\partial t} = -k^2Be^{(ik\cdot\bar{x} + \lambda t)} - sBe^{(ik\cdot\bar{x} + \lambda t)} + \frac{s}{(1 + N)^2} Ae^{(ik\cdot\bar{x} + \lambda t)}
\]
Simplifying:

\[
\begin{align*}
\frac{\partial n_1}{\partial t} &= -Dk^2 n_1 - srNn_1 + aNk^2 c_1 \\
\frac{\partial c_1}{\partial t} &= -k^2 c_1 - sc_1 + \frac{s}{(1+N)^2} n_1
\end{align*}
\]

We now write this in matrix form:

\[
\frac{d}{dt} \begin{pmatrix} n_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} -Dk^2 - srN - \lambda & aNk^2 \\ \frac{s}{(1+N)^2} & -k^2 - S - \lambda \end{pmatrix} \begin{pmatrix} n_1 \\ c_1 \end{pmatrix}
\]

We want to solve this set of equations to determine the linearised response to a small perturbation of the given form from the steady state \(c_0, n_0\). We therefore have:

\[
0 = \det \begin{pmatrix} -Dk^2 - srN - \lambda & aNk^2 \\ \frac{s}{(1+N)^2} & -k^2 - S - \lambda \end{pmatrix}
= (-Dk^2 - srN - \lambda)(-k^2 - S - \lambda) - \frac{s}{(1+N)^2} aNk^2
= (Dk^2 + srN + \lambda)(k^2 + S + \lambda) - \frac{s}{(1+N)^2} aNk^2
= (Dk^2k^2 + srNk^2 + \lambda k^2 + Dk^2s + srNs + \lambda s + Dk^2\lambda + srN\lambda + \lambda^2) - \frac{s}{(1+N)^2} aNk^2
= \lambda^2 + (Dk^2k^2 + srNk^2 + Dk^2s + srNs + \lambda k^2 + \lambda s + Dk^2\lambda + srN\lambda - \frac{s}{(1+N)^2} aNk^2
= \lambda^2 + \lambda(k^2 + s + Dk^2 + srN) + (Dk^2k^2 + srNk^2 + Dk^2s + srNs) - \frac{s}{(1+N)^2} aNk^2
0 = \lambda^2 + \lambda((D + 1)k^2 + s(1 + rN)) + (k^2 + s)(Dk^2 + srN) - \frac{s}{(1+N)^2} aNk^2
\]

This dispersion relation describes how the wavenumber \(k\) varies with the frequency \(\lambda\). The perturbations will grow over time if \(Re(\lambda) > 0\) and will decay if \(Re(\lambda) < 0\), so we are interested in the stability cutoff where \(\lambda = 0\). This leads us to the equation:

\[
0 = (k^2 + s)(Dk^2 + srN) - \frac{s}{(1+N)^2} aNk^2
0 = Dk^4 + \left[ srN + sD - \frac{s}{(1+N)^2} \right] k^2 + s^2 rN
\]

Thus we have determined that the uniform equilibrium state is \textit{unstable} in the presence of diffusion only if there exists a wavenumber \(k\) for which the above quantity is \textit{less than zero}, as then \(\lambda\) must be greater than 0 in order for the dispersion relation to hold (as it always does).

We now set \(\alpha = \left(2r + \frac{D}{N}\right)(1+N)^2\) as a simple special case, and consider the conditions under which pattern formation develops. The dispersion equation becomes:

\[
\begin{align*}
0 = \lambda^2 + \lambda((D + 1)k^2 + s(1 + rN)) + (k^2 + s)(Dk^2 + srN) - \frac{s}{(1+N)^2} \left(2r + \frac{D}{N}\right)(1+N)^2 Nk^2 \\
0 = \lambda^2 + \lambda((D + 1)k^2 + s(1 + rN)) + (k^2 + s)(Dk^2 + srN) - (2rsNk^2 + Dsk^2)
\end{align*}
\]
0 = \lambda^2 + \lambda((D + 1)k^2 + s(1 + rN)) + Dk^4 + s^2rN - rsNk^2
0 = \lambda^2 + \lambda((D + 1)k^2 + s(1 + rN)) + Dk^4 + s^2rN - rsNk^2

We thus have the condition that the uniform equilibrium state is unstable to perturbations with values of \( k \) for which:

\[ Dk^4 + s^2rN - rsNk^2 < 0 \]

We can show that this is equivalent to the condition in the question by working backwards:

\[
\left| \frac{2Dk^2}{srN} - 1 \right| < \left(1 - \frac{4D}{rN}\right)^\frac{1}{2}
\]
\[
\left( \frac{2Dk^2}{srN} - 1 \right)^2 < \left(1 - \frac{4D}{rN}\right)
\]
\[
\left( \frac{2Dk^2}{srN} \right)^2 - 4Dk^2 + 1 < 1 - \frac{4D}{rN}
\]
\[
\frac{4D^2k^4}{srN} - 4Dk^2 < -4sD
\]
\[
Dk^4 - ks^2rN < -s^2rN
\]
\[
Dk^4 + s^2rN - rsNk^2 < 0
\]

We thus have the condition that pattern formation occurs values of \( k \) that satisfy:

\[
\left| \frac{2D}{srN} k^2 - 1 \right| < \left(1 - \frac{4D}{rN}\right)^\frac{1}{2}
\]

If we take the further special case of \( \frac{4D}{rN} = 3/4 \) and measure distance in units of \( \frac{2D}{srN} \), this condition becomes simply:

\[
|k^2 - 1| < \frac{1}{2}
\]
\[
-\frac{1}{2} < k^2 - 1 < \frac{1}{2}
\]
\[
\frac{1}{2} < k^2 < \frac{3}{2}
\]

To convert from a length to wavenumber use \( k = 2\pi/\lambda \). For fixed boundaries in both dimensions at both ends, we will have \( \lambda = (2/n)L \). Hence:

\[
k_x = \frac{2\pi}{(2/n)\L_x} = \frac{n\pi}{\L_x}
\]

If we have \( (\L_x, \L_y) = (4,4) \) then:

\[
k^2 = \left( \frac{n_x\pi}{\L_x} \right)^2 + \left( \frac{n_y\pi}{\L_y} \right)^2
\]
\[
k^2 = \left( \frac{n_x\pi}{4} \right)^2 + \left( \frac{n_y\pi}{4} \right)^2
\]
\[
k^2 = \left( \frac{n_x\pi}{4} \right)^2 + \left( \frac{n_y\pi}{4} \right)^2
\]
In the $n_x = n_y = 1$ case:

\[
\begin{align*}
  k^2 &= 2 \left( \frac{\pi}{4} \right)^2 \\
  k^2 &\approx 2 \left( \frac{9}{16} \right) \\
  k^2 &= \frac{9}{8}
\end{align*}
\]

**Useful Mathematics**

**Series**

\[
\begin{align*}
  f(x) &= f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \cdots \\
  e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
  \frac{1}{1-x} &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
  \frac{1}{1+x} &= \sum_{n=0}^{\infty} (-x)^n = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \\
  \frac{1}{1-(z+2)^2} &= \sum_{n=0}^{\infty} (z+2)^{2n} \\
  \frac{1}{1+(z+2)^3} &= \sum_{n=0}^{\infty} (-1)^n(z+2)^{3n}
\end{align*}
\]

**Trigonometry**

\[
\begin{align*}
  \sin(2\theta) &= 2 \sin \theta \cos \theta \\
  \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 \\
  \tan(2\theta) &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \\
  \sin^2 \theta &= \frac{2}{1 - \cos 2\theta} \\
  \cos^2 \theta &= \frac{1 + \cos 2\theta}{2} \\
  \cos x &= \frac{e^{ix} + e^{-ix}}{2} \\
  \sin x &= \frac{e^{ix} - e^{-ix}}{2i}
\end{align*}
\]

arcsin : $[-1, 1] \rightarrow \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$

arccos : $[-1, 1] \rightarrow [0, \pi]$

arctan : $\mathbb{R} \rightarrow \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$
Delta function

\[ \int_{-\infty}^{\infty} \delta(ax) \, dx = \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(x) \, dx \]

\[ \int_{-\infty}^{\infty} f(t) \delta(t-T) \, dt = f(T) \]

\[ \int_{-\infty}^{\infty} \delta(\xi-x) \delta(x-\eta) \, dx = \delta(\xi-\eta) \]

Fourier theorems

\[ \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx \]

\[ f(x = a) = e^{-i\omega a} \hat{f}(\omega) \]

\[ f(ax) = \frac{1}{|a|} \hat{f} \left( \frac{\omega}{a} \right) \]

\[ \frac{d^n f(x)}{dx^n} = (i\omega)^n \hat{f}(\omega) \]

\[ x^n f(x) = i^n \frac{d^n}{d\omega^n} \hat{f}(\omega) \]

\[ h(x) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) \, d\tau \rightarrow \hat{h}(\omega) = \hat{f}(\omega) \hat{g}(\omega) \]

\[ \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{x}(\omega)|^2 \, d\omega \]

Fourier transforms

\[ e^{-ax^2} \rightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4a}} \]

\[ 1 \rightarrow \sqrt{2\pi} \delta(\omega) \]

\[ \delta(x) \rightarrow \frac{1}{\sqrt{2\pi}} \]

\[ e^{i\alpha x} \rightarrow \sqrt{2\pi} \delta(\omega-a) \]

\[ \cos(ax) \rightarrow \frac{\sqrt{2\pi}}{2} \left( \delta(\omega-a) + \delta(\omega+a) \right) \]

\[ \sin(ax) \rightarrow \frac{\sqrt{2\pi}}{2i} \left( \delta(\omega-a) - \delta(\omega+a) \right) \]

\[ x^n \rightarrow i^n \sqrt{2\pi} \delta^n(\omega) \]

\[ \delta^n(x) \rightarrow \frac{(i\omega)^n}{\sqrt{2\pi}} \]

\[ \frac{1}{x} \rightarrow -i \frac{\sqrt{2\pi}}{2} \text{sgn}(\omega) \]

Complex analysis

\[ \text{Res}(f; a) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)] \]

\[ \oint_{\Gamma} f(z) \, dz = 2\pi i \sum_{k=1}^{m} \text{Res}(f; z_k) \]
Ordinary differential equations

\[ w'(t) - aw(t) = 0 \rightarrow w(t) = a \int w(t) \, dt \]
\[ au''(t) + bu'(t) + cu(t) = 0 \rightarrow w(t) = e^{at} \]
\[ w'(t) + p(t)w(t) = r(t) \rightarrow I(t) = e^{\int p(t) \, dt} \rightarrow \frac{d(I(t)w(t))}{dt} = I(t) \]

\[ y_{gen}(x) = y_{homo}(x) + y_{part}(x) \]

<table>
<thead>
<tr>
<th>Term in ( r(x) )</th>
<th>Choice for ( y_p(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ke^{\gamma x} )</td>
<td>( C_0 e^{\gamma x} )</td>
</tr>
<tr>
<td>( k\chi^n (n = 0, 1, \ldots) )</td>
<td>( K_0 e^{\chi_1 x} + K_{n-1} e^{\chi_{n-1} x} + \cdots + K_1 e^{\chi_1 x} )</td>
</tr>
<tr>
<td>( k\cos \omega x )</td>
<td>( K\cos \omega x + M\sin \omega x )</td>
</tr>
<tr>
<td>( k\sin \omega x )</td>
<td>( e^{i\omega x} (K\cos \omega x + M\sin \omega x) )</td>
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<td>( e^{i\omega x} (K\cos \omega x + M\sin \omega x) )</td>
</tr>
</tbody>
</table>

The Cauchy-Euler equation

\[ at^2w''(t) + \beta tw'(t) + \gamma w(t) = 0 \rightarrow w(t) = t^\mu \]

- Case #1: Two distinct roots, \( m_1 \) and \( m_2 \)
- Case #2: One real repeated root, \( m \)
- Case #3: Complex roots, \( \alpha \pm \beta i \)

In case #1, the solution is given by:

\[ y = c_1 x^{m_1} + c_2 x^{m_2} \]

In case #2, the solution is given by

\[ y = c_1 x^m \ln(x) + c_2 x^m \]

Integration by parts

\[ \int u(x) \frac{dv(x)}{dx} \, dx = u(x)v(x) - \int \frac{du(x)}{dx} v(x) \, dx \]
\[ \iint f \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dxdy = \int fP \, dx + \int fQ \, dy - \iint \left( \frac{\partial f}{\partial x} Q - \frac{\partial f}{\partial y} P \right) \, dxdy \]